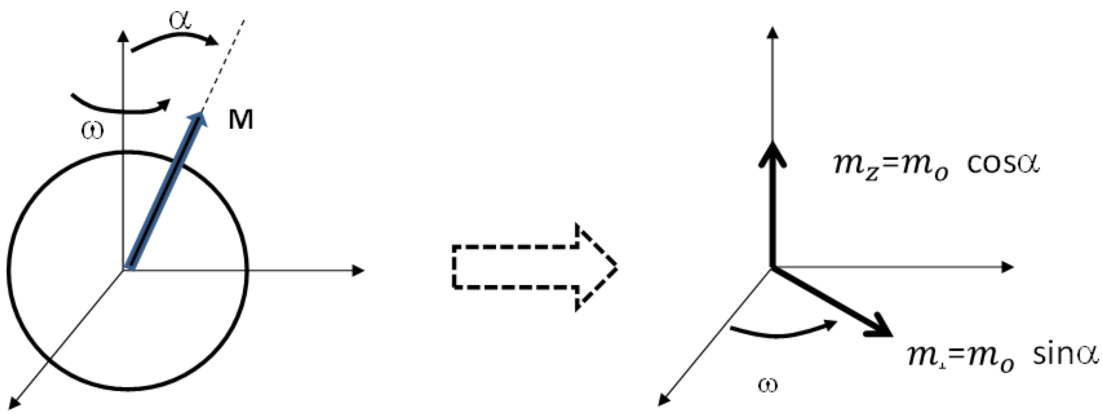
Favorite Physics Problems

Electricity and Magnetism {E&M}

A sphere is uniformly magnetized and is rotating at an angular velocity ω about an axis through its center and the angle between the rotation axis and the magnetization **M** axis is α. The rotating magnetized sphere is radiating an electromagnetic wave. Find the electromagnetic field vectors of the radiated field. (see Hauser, Walter, “Introduction to the Principles of Electromagnetism”, Addison-Wesley Pub., 1971, page 420, problem # 13.7).



PROLOGUE

Two different methods, shown here under Parts I and II, are used to solve this problem.

In Part I, a detailed lengthy method is described that begins with Hauser’s equation 13.105, i.e., it begins from “scratch”. To avoid the problem of HAROLD, *(i.e.,Hypothetical Alert Reader of Limitless Dedication: as found in Julian Schwinger's "Particles, Sources and Fields" of 1970), many steps are shown in Part 1 which might seem unnessary to advanced readers. However, it is easier for such readers to ignore the details than for a struggling reader to perform the magic to fill in the details. (It is certainly recognized that the struggle of filling in the details is a good learning experience but, on the other hand, seeing the details can be a useful approach to learning techniques.)*

In Part 2, the solution takes an "easier and shorter" path by beginning with Hauser's equation 13.108 and applying the results from Hauser's problem 13.4. Part II shows the application of a result from another problem but it is understood that the solution of problem 13.4 is rather complicated. But the point is to demonstrate how a solution of a rotating electric dipole (i.e., the essence of problem 13.4) can be applied to the case of a rotating magnetization through Hauser's equation 13.108. That is why Part II was noted as "easier and shorter. It is only "easier and shorter" because the solution to another useful problem is already in hand. Part II helps a reader to appreciate the complicated connections in different types of problems that often can be used in practice to help acheive a solution to a particular issue in science or engineering.

PART I

So, to begin this problem first look at two vectors:
$m_z = m_0 \cos\alpha$ is a static magnetic dipole and will not radiate
$m_\perp$ is a rotating magnetic dipole and will radiate.

We'll use a common trick and that is we'll represent the rotating magnetic dipole, $m_\perp$, in the x-y plane, as the sum of two perpendicular (i.e., $\perp$) sinusoidally varying magnetic dipoles, in the x-y plane,and are $\pi/2$ radians out of phase with one another. And that is reprsented by Equation 1.

$\boldsymbol{m}_{\perp}= m_0 \sin\alpha\cos\omega t\ \hat{\boldsymbol{i}} + m_0 \sin\alpha \sin\omega t\ \hat{\boldsymbol{j}}$

(1)

let $\boldsymbol{m_1} = m_0 \sin\alpha\, e^{-i\omega t}\ \hat{\boldsymbol{i}}$

and $\boldsymbol{m_2} = m_0 \sin\alpha\, e^{-i\left(\omega t-\frac{\pi}{2}\right)} \hat{\boldsymbol{j}} = m_0 \sin\alpha\, e^{-i\omega t} \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) \hat{\boldsymbol{j}}$

$= m_0 \sin\alpha\, i\, e^{-i\omega t}\, \hat{\boldsymbol{j}}$

we now have $\boldsymbol{m}_{\perp}= m_0 \sin\alpha\, (\hat{\boldsymbol{i}}+i\hat{\boldsymbol{j}})\, e^{-i\omega t}$

(2)

we note that the real part of equation (2) is the same as equation (1).

$\boldsymbol{A}^{(2)}(\mathbf{r},t)= \frac{\mu_0}{4\pi} \nabla\times\left[\frac{m(t-r/c)}{r}\right]$ see (Hauser’s equation13.105)

$\mathbf{m}(t') = \boldsymbol{m_0}\, e^{-i\omega t'} = \boldsymbol{m_0}\, e^{-i\omega(t-r/c)}$

or, with dropping of ‘, $\mathbf{m}(t) = \boldsymbol{m_0}\, e^{-i(\omega t-kr)} = \boldsymbol{m_0}\, e^{-i(kr-\omega t)}$, then

$\boldsymbol{A}^2(\mathbf{r},t) = \frac{\mu_0}{4\pi} \nabla\times\left[\frac{\boldsymbol{m_0}\, e^{-i(kr-\omega t)}}{r}\right]$

$= \frac{\mu_0}{4\pi} \nabla\times\left[\frac{\boldsymbol{m_0}\, e^{i(kr)}}{r}\right] e^{-i\omega t}$

Using the identity $\nabla\times(f\mathbf{B}) = (\nabla f) \times \mathbf{B} + f\,(\nabla\times\mathbf{B})$,

take $f = \frac{e^{i(kr)}}{r}$ and $\mathbf{B} = \boldsymbol{m_0}$ where $\boldsymbol{m_0}$ is a constant vector then $\nabla\times[\frac{\boldsymbol{m_0}\, e^{i(\mathrm{kr})}}{r}] = \nabla\frac{e^{i(\mathrm{kr})}}{r}\times\boldsymbol{m_0} + \frac{e^{i(\mathrm{kr})}}{r}\nabla\times\boldsymbol{m_0}$ and since $\boldsymbol{m_0}$ is a constant vector the last term is 0.

$$\therefore \boldsymbol{A}^{(2)}(\mathbf{r},\mathrm{t}) = \frac{\mu_0}{4\pi}\,\nabla\left(\frac{e^{i\mathrm{kr}}}{r}\right)\times\boldsymbol{m_0}e^{-i\omega\mathrm{t}}$$

(3)

To evaluate $\nabla\left(\frac{e^{i\mathrm{kr}}}{r}\right)$ recall that $\nabla\phi=\frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\vartheta}\hat{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$ and use it to evaluate $\nabla\left(\frac{e^{i\mathrm{kr}}}{r}\right)$ as

$$\nabla\left(\frac{e^{i\mathrm{kr}}}{r}\right) = \frac{\partial}{\partial r}\frac{e^{i\mathrm{kr}}}{r}\,\hat{e}_r + 0 + 0.$$

(4)

The right hand side of equation (4) has the appearance of a spherical Hankel function where the spherical Hankel function of the first kind of order ℓ are defined by the equation:

$h_\ell^{(1)}(\mathrm{kr}) = -i(-1)^\ell\,(\mathrm{kr})^\ell\left(\frac{1}{\mathrm{kr}}\frac{\partial}{\partial\mathrm{kr}}\right)^\ell\left(\frac{e^{i\mathrm{kr}}}{\mathrm{kr}}\right)$ and with ℓ=1 Spherical Hankel function becomes:

$$h_1^{(1)}(\mathrm{kr}) = i(\mathrm{kr})\left[\frac{1}{\mathrm{kr}}\frac{\partial}{\partial\mathrm{kr}}\left(\frac{e^{i\mathrm{kr}}}{\mathrm{kr}}\right)\right]$$

$$= \frac{i}{\mathrm{k}^2}\,\frac{\partial}{\partial r}\frac{e^{i\mathrm{kr}}}{r}\ .$$

(5)

Using equations (4) and (5) in equation (3) gives

$\boldsymbol{A}^{(2)}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\ \frac{k^2}{i}\left(h_1^{(1)}(kr)\,\hat{e}_r\right) \times \boldsymbol{m_0} e^{-i\omega t}$.

In our case with $\boldsymbol{m}_\perp = m_0 \sin\alpha\ (\hat{\boldsymbol{i}} + i\hat{\boldsymbol{j}})\ e^{-i\omega t}$, $\boldsymbol{m_0} \longrightarrow m_0 \sin\alpha$ $(\hat{\boldsymbol{i}} + i\hat{\boldsymbol{j}})$

$\therefore \boldsymbol{A}^{(2)}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\ \frac{k^2}{i}\left(h_1^{(1)}(kr)\,\hat{e}_r\right) \times (\hat{\boldsymbol{i}} + i\hat{\boldsymbol{j}})\ m_0 \sin\alpha\ e^{-i\omega t}$.
(6)

Now consider the quantity $\left(h_1^{(1)}(kr)\,\hat{e}_r\right) \times (\hat{\boldsymbol{i}} + i\hat{\boldsymbol{j}})$ from equation (6). First recall that
$\nabla x = \frac{\partial x}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial x}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial x}{\partial z}\hat{\boldsymbol{k}} = \hat{\boldsymbol{i}}$ and going back to $\left(h_1^{(1)}(kr)\,\hat{e}_r\right) \times (\hat{\boldsymbol{i}} + i\hat{\boldsymbol{j}})$ select the first term from the cross-product, namely, and substitute for $\hat{\boldsymbol{i}}$:

$h_1^{(1)}(kr)\,\hat{e}_r \times \hat{\boldsymbol{i}} = h_1^{(1)}(kr)\,\hat{e}_r \times \nabla x$. Next convert the x to spherical coordinates to obtain

$h_1^{(1)}(kr)\,\hat{e}_r \times \hat{\boldsymbol{i}} = h_1^{(1)}(kr)\,\hat{e}_r \times \nabla(r\cos\varphi\sin\vartheta)$ in spherical coordinates. Recall the identity

$\nabla \times f = \nabla f + f(\nabla \times \boldsymbol{B})$ and let $f = (r\cos\varphi\sin\vartheta)$ and $\boldsymbol{B} = h_1^{(1)}(kr)\,\hat{e}_r$ then, with a little rearrangement of terms in the identity,

∇(rcosφsinϑ) × $h_1^{(1)}$(kr) $\hat{e}_r$ = -∇×(rcosφsin$\vartheta h_1^{(1)}$(kr) $\hat{e}_r$) - (rcosφsinϑ)(∇×$h_1^{(1)}$(kr) $\hat{e}_r$)
and the last term is zero as can be seen by considering:

$\nabla \times h_1^{(1)}(\mathrm{kr})\, \hat{e}_r = \begin{pmatrix} \frac{\hat{e}_r}{r^2 \sin\vartheta} & \frac{\hat{e}_\vartheta}{\mathrm{r}\sin\vartheta} & \frac{\hat{e}_\varphi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \vartheta} & \frac{\partial}{\partial \varphi} \\ h_1^{(1)} & 0 & 0 \end{pmatrix}$ = 0. So now looking at the remaining term,

-∇×(rcosφsin$\vartheta h_1^{(1)}$(kr) $\hat{e}_r$), and recognizing that r $\hat{e}_r$ = **r**, the term involving "×$\hat{\boldsymbol{i}}$" becomes

$h_1^{(1)}$(kr) $\hat{e}_r$× $\hat{\boldsymbol{i}}$= -∇×(**r**cosφsin$\vartheta h_1^{(1)}$(kr)) . (7)

For the remaining "×$\hat{\boldsymbol{j}}$" term, i.e., $\left(h_1^{(1)}(\mathrm{kr})\, \hat{e}_r\right)$×$i\hat{\boldsymbol{j}}$, steps similar to those used above are followed t obtain a similar result. First, regognize that ∇y = $\hat{\boldsymbol{j}}$ and that y = rsinφcosϑ then after placing i out front,

$i\left(h_1^{(1)}(\mathrm{kr})\, \hat{e}_r\right)$×$i\hat{\boldsymbol{j}}$=$h_1^{(1)}$(kr) $\hat{e}_r$× ∇(rsinφcosϑ). Using the same identity again,

i.e., ∇×f = ∇f + f(∇×***B***) , substitution provides the following result:

∇(rsinϑsinϑ) × $h_1^{(1)}$(kr) $\hat{e}_r$ = ∇×(rsinφsin$\vartheta h_1^{(1)}$(kr) $\hat{e}_r$) - (rsinφsinϑ)(∇×$h_1^{(1)}$(kr) $\hat{e}_r$). As shown earlier, the last term is “0” and using r $\hat{e}_r$ = **r**, the remainding expression becomes:

$$i h_1^{(1)}(\text{kr})\, \hat{e}_r \times \hat{\boldsymbol{j}} = -i \nabla\times(\mathbf{r}\sin\varphi\sin\vartheta\, h_1^{(1)}(\text{kr})). \tag{8}$$

Using equations (7) and (8) in equation (6) results in the following equation:

$$\boldsymbol{A}^{(2)}(\mathbf{r},\text{t}) = \frac{\mu_0}{4\pi} \frac{\text{k}^2}{i} m_0 \sin\alpha\, e^{-i\omega \text{t}} \left\{-\nabla\times\left(\boldsymbol{r}\cos\varphi\sin\vartheta\, h_1^{(1)}(\text{kr})\right) - i\,\nabla\times\left(\boldsymbol{r}\sin\varphi\sin\vartheta\, h_1^{(1)}(\text{kr})\right)\right\}.$$

The terms in the curley brackets, abbreviated as {■} here, can be simplified as shown in the following steps:

$$\{■\} = -\nabla\times(\boldsymbol{r}\cos\varphi\sin\vartheta\, h_1^{(1)}(\text{kr}) - i\boldsymbol{r}\sin\varphi\sin\vartheta\, h_1^{(1)}(\text{kr}))$$
$$= -\nabla\times[\boldsymbol{r}\sin\vartheta\, h_1^{(1)}(\text{kr})\,(\cos\varphi + i\sin\varphi)]$$
$$= -\nabla\times[\boldsymbol{r}\sin\vartheta\, h_1^{(1)}(\text{kr})\, e^{i\varphi}]$$

Upon substituting the last quantity into the above equation for $\boldsymbol{A}^{(2)}$(**r**,t), the expression for $\boldsymbol{A}^{(2)}$(**r**,t) becomes:

$\boldsymbol{A}^{(2)}(\mathbf{r},\mathrm{t})= -\frac{\mu_0}{4\pi}\,\frac{\mathrm{k}^2}{i}m_0\sin\alpha\, e^{-i\omega \mathrm{t}}\left\{\nabla\times\left[\boldsymbol{r}\sin\vartheta\, h_1^{(1)}(\mathrm{kr})\, e^{i\varphi}\right]\right\}$ and this can be placed into the

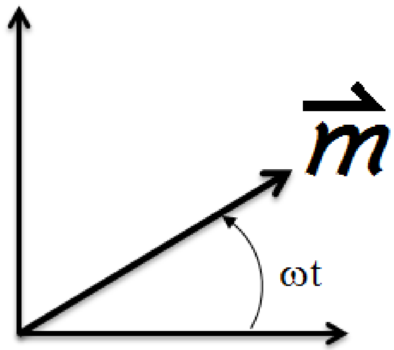
following form:

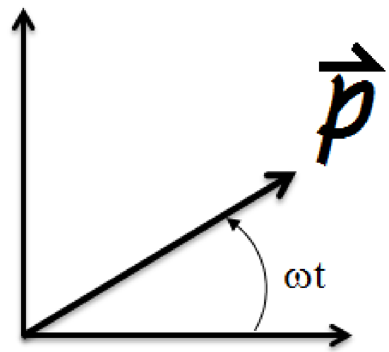
$$\boldsymbol{A}^{(2)}(\mathbf{r},\mathrm{t}) = \frac{i\,\mathrm{k}^2\,\mu_0\, m_0 \sin\alpha}{4\pi}\;\nabla\times\left[\boldsymbol{r}\sin\vartheta e^{i\varphi}\, h_1^{(1)}(\mathrm{kr})\,\right] e^{-i\omega \mathrm{t}}\;. \tag{9}$$

Equation (9) is the final result and it agrees with the answer given by Hauser on his page 609 for his problem 13.7.

PART II

As noted in the Prologue, a second method is offer in this part to solve the problem of interest. It stems from a certain aspect of a rotating magnetic dipole, $\vec{m}$, and rotating electric current moment, $\vec{p}$, with both rotating at the same angular frequency ω.





Specifically, this approach begins with Hauser's equation 13.108 which describes the interesting fact that the magnetic vector potential, $\boldsymbol{A}(\mathbf{r},t)$, of a magnetic dipole $\vec{m}$ has "the same spatial dependence as the magnetic induction field" $\boldsymbol{B}^{e.d.}$, of an electric current moment, $\vec{p}$ (see Hauser pages 396-397):

$$\boldsymbol{A}^{(2)}(\mathbf{r},t) = -\frac{i m_0}{p_0\,\omega}\,\boldsymbol{B}^{e.d.}. \qquad (10)$$

The solution, which is lengthly, to Hauser's problem 13.4 provides an expression for a rotating electric dipole as:

$\boldsymbol{B}^{e.d.}(\mathbf{r},t)= -\frac{\mu_0}{4\pi}\omega k^2 p_0 \sin\alpha\, e^{-i\omega t}\left\{\nabla\times\left[\boldsymbol{r}\sin\vartheta\, h_1^{(1)}(kr)\, e^{i\varphi}\right]\right\}$. Using this expression in equation

(10) with m_0 replaced with $m_0\sin\alpha$ results in:

$$\boldsymbol{A}^{(2)}(\mathbf{r},t) = \frac{i m_0 \sin\alpha}{p_0\,\omega}\,\frac{\mu_0}{4\pi}\omega k^2 p_0 \sin\alpha\, e^{-i\omega t}\left\{\nabla\times\left[\boldsymbol{r}\sin\vartheta\, h_1^{(1)}(kr)\, e^{i\varphi}\right]\right\}$$

$$= \frac{i\, k^2\, \mu_0\, m_0 \sin\alpha}{4\,\pi}\; \nabla \times \left[\boldsymbol{r} \sin\vartheta\, e^{i\varphi}\, h_1^{(1)}(\mathrm{kr})\right] e^{-i\omega \mathrm{t}}\;,$$

(11)

which is the same equation found in Part I.